The Determination of Eros' Parallax
Mathematical Details

U. Backhaus, Universität Duisburg-Essen, Germany

In order to measure the distance of an object from the earth we measure its *topocentric* equatorial coordinates from two different sites on earth. To determine the point of intersection of the two lines of view we take use of vector calculus.

1 The algorithm

We transform all of the spherical vectors in rectangular vectors with respect to the coordinate system the x-axis and z-axis of which are pointing toward the vernal equinox and the celestial north pole, respectively. The y-axis is oriented so that the system forms a "'right hand system"'. The direction of the y-axis is therefore (α=6h, $\delta = 0°$).

1. The measured topocentric positions (α_i, δ_i) give us the directions of the lines of view. We transform them into normalized rectangular vectors (see Fig. 2):

$$\vec{e}_i = (\cos\alpha_i \cos\delta_i,\ \sin\alpha_i \cos\delta_i,\ \sin\delta_i) \tag{1}$$

   (You can easily prove that these vectors are of unit length!)

2. In order to calculate the intersection of the two lines of view we need the equatorial positions of the two observers:

   (a) Figure 1 shows an observer and the vernal equinox just laying in the same plane.

   This picture shows you: **The declination of an observer is equal to his geographical latitude.** (For good results we will have to take into account that the earth's body is not a perfect sphere but an ellipsoid with excentricity e = 0.08182 (see, for instance O. Montenbruck et al: Astronomy on the personal computer, Springer, New York 1994). For simplicity, our algorithm does not take into account this refinement.

   (b) At the moment of Figure 2, the right ascension of the observer is obviously $\alpha = 0h$. The vernal equinox is just culminating in the observer's horizontal system. Therefore, his local sideral time is 0h. Due to the earth's daily revolution the right ascension of the observer will increase by a rate of 24h in 23h56min (the earth's period of rotation) with exactly the same rate as his local sideral time!

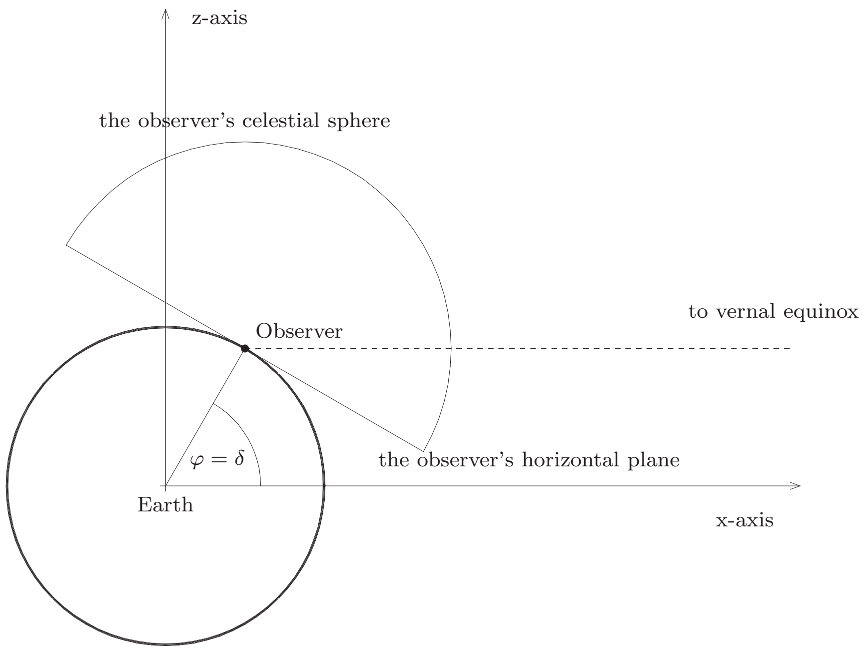

Figure 1: Determination of the geocentric equatorial coordinates of an observation site: At 0.00h local sideral time, when the vernal equinox culminates, the right ascension of the site is 0.00h, too.

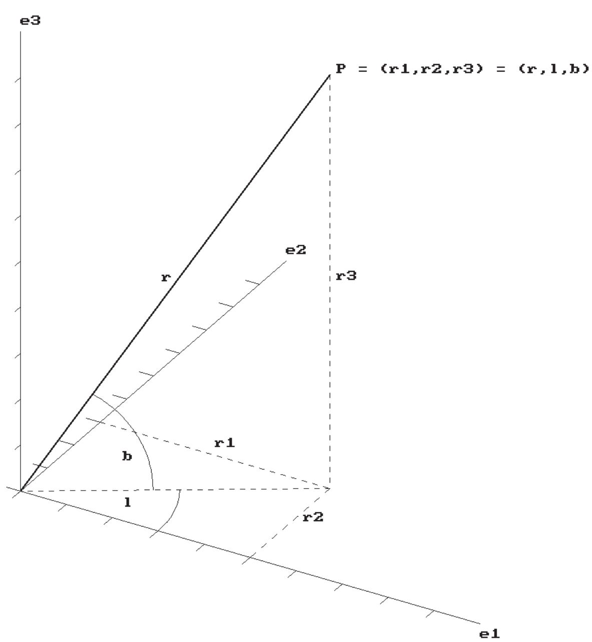

Figure 2: The relation between polar coordinates (r, l, b) and rectangular coordinates $(r_1.r_2, r_3)$

That means: **The right ascension of an observer is equal to his local sideral time.**

(c) There are two possibilities to determine local sideral time:

i. From an astronomical almanach you can get the sideral time $st_0(Gr)$ of the day[1] for 0h UT at Greenwich longitude. At this time the local sideral time $st_0(Obs)$ of an observer is

$$st_0(Obs) = st_0(Gr) + l\frac{4\ min}{1^\circ} \tag{2}$$

(l = geographical longitude of the observer, east of Greenwich taken as positive).
At the time t the observer's local sideral time is

$$st(Obs) = st_0(Obs) + 1.0027379t \tag{3}$$

The factor in this equation is due to the fact that sideral time increases by 24h in 23h56min approximately.

ii. Today you can easily get a computer algorithm for calculating local sideral time (see our program, for instance).

3. The sperical coordinates $(\alpha_{O_i}, \delta_{O_i})$ of the observers can be transformed into rectangular coordinates as the asteroid's coordinates:

$$\vec{r}_{O_i} = \rho_i(\cos\alpha_{O_1}\cos\delta_{O_i},\ \sin\alpha_{O_i}\cos\delta_{O_i},\ \sin\delta_{O_i}) \tag{4}$$

(ρ_i is approximately the earth's equatorial radius but, more exactly, the geocentric distance of the observer expressed as a multiple of the earth's equatorial radius).

4. Now we are able to determine the point of intersection of the two lines of view:

$$\vec{r}_{O_1} + \lambda\vec{e}_1 = \vec{r}_{O_2} + \mu\vec{e}_2$$

These are three equations with only two unknown quantities. Unfortunately, due to measuring errors the two lines usually will not intersect and the above system will not have any solution!

5. For this reason, we are forced to calculate the closest distance of the respective lines instead of their point of intersection. That means we are looking for two points P_1 and P_2 (or two vectors $\vec{r}_1$ and $\vec{r}_2$), respectively

$$\vec{r}_1 = \vec{r}_{O_1} + \lambda\vec{e}_1,\ \vec{r}_2 = \vec{r}_{O_2} + \mu\vec{e}_2,$$

[1] We use January, 28th, 2012

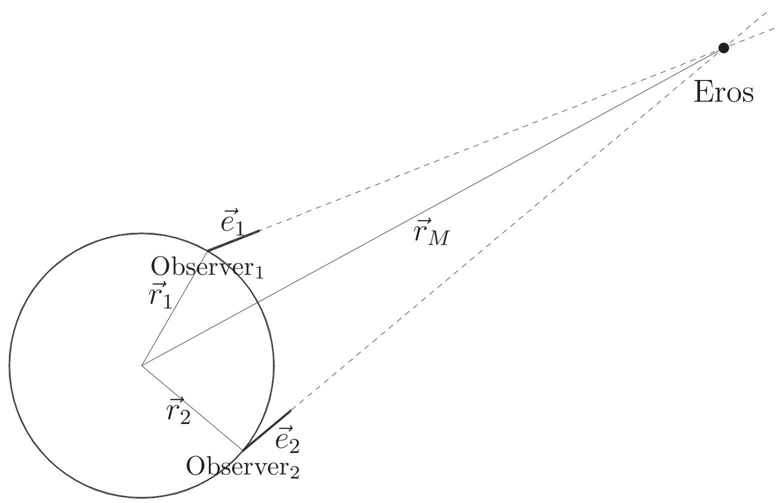

Figure 3: The two lines of view intersect at the asteroid. Taking measurement errors into account the distance of the two lines will there be minimal.

so that the vector connecting these points is perpendicular to the respective directions of view:

$$\begin{aligned}(\vec{r}_1 - \vec{r}_2) \cdot \vec{e}_1 &= 0,\\ (\vec{r}_1 - \vec{r}_2) \cdot \vec{e}_2 &= 0\end{aligned}$$

This is a system of two linear equations with two unknown quantities. Therefore, the problem has a unique solution.

Try to find the following solution by yourself:

$$\lambda + \mu = \frac{(\vec{r}_{O_2} - \vec{r}_{O_1}) \cdot (\vec{e}_2 - \vec{e}_1)}{1 - \vec{e}_1 \cdot \vec{e}_2} \tag{5}$$

$$\lambda - \mu = \frac{(\vec{r}_{O_2} - \vec{r}_{O_1}) \cdot (\vec{e}_2 + \vec{e}_1)}{1 + \vec{e}_1 \cdot \vec{e}_2} \tag{6}$$

Now it is easy to calculate λ and μ:

$$\lambda = \frac{1}{2}\left((\lambda + \mu) + (\lambda - \mu)\right), \tag{7}$$

$$\mu = \frac{1}{2}\left((\lambda + \mu) - (\lambda - \mu)\right) \tag{8}$$

6. The geocentric distance d_P of the asteroid can now be calculated easily:

$$d_P \approx |\vec{r}_{O_1} + \lambda \vec{e}_1| \approx |\vec{r}_{O_2} + \mu \vec{e}_2| \tag{9}$$

2 Concretisation of the algorithm in the spread sheet "'erosparallax.xls"'

The algorithm described above is implemented into the spreadsheet `"erosparallax.xls"`.

- The input data of the locations of interest are copied into the rows 7 and 8 of the input work sheet.
- The steps of the algorithm are done in the worksheet "'Calculation"'. They are located in the following cells:

Cells B4-B13 Transfer of the input data. For the position data a complicated conversion is necessary.

Cells E4-E9 Calculation of the both rectangular coordinates of Eros corresponding to equation (1)

Cells E10-E11 Calculation of the both observation times in days with decimals.

Cells E12-E13 Calculation of the both local sideral times in hours with decimals corresponding to equations (2) and (3).

Cells E14-E19 Calculation of the rectangular equatorial coordinates of the both observers due to equation (4).

Cell H4 The scalar product $\vec{e}_1 \times \vec{e}_2$ of the both directions in equations (5) and (6)

Cells H5-H6 Calculation of $\lambda + \mu$ and $\lambda - mu$ due to equations (5) and (6).

Cells E14-E19 Calculation of λ und μ due to (7) and (8).

Cells H10-H11 Calculation of the both geocentric distances of Eros due to equation (9).

Cell H13 Calculation of the minimal distance of the two lines of view due to the following equation:

$$\Delta = |\vec{r}_1 - \vec{r}_2|. \tag{10}$$

- Finally, the results are transferred to worksheet "'Input and results"':

Cells B13-C13 The geocentric distance of Eros as the arithmetical mean of the both results:

$$\boldsymbol{d_P = \frac{1}{2}\left(|\vec{r}_{O_1} + \lambda\vec{e}_1| + |\vec{r}_{O_2} + \mu\vec{e}_2|\right)} \tag{11}$$

Cells D13-E13 Eros' calculated distance as determined by the Jet Propulsion Laboratory (`http://ssd.jpl.nasa.gov/horizons.cgi?s_time=1#top`). Because these distances are contained in worksheet "'Distance of Eros"' only for full hours, this value may be replaced by the formula of the fit line (column D).

Cells D13-E13 As the distance is known as multiples of R_E and AU the Astronomical Unit AU can be calculated as a multiple of R_E or in kilometers if the earth's radius is known (cell B30 in "'Calculation"').

Cell C14 The minimal distance of the both lines of view may be taken as a measure for the quality of the both measurements.