On the transit of a planet across the sun

by Robert Stawell Ball

1 Introductory

If the orbit of Venus were in the plane of the ecliptic then whenever the geocentric longitude of Venus was the same as that of the sun the planet would appear near the centre of the solar disc. About three hours previously the terrestrial observer would have seen the planet entering on the suns disc, about three hours later the planet would pass off from the disc and during the six hours of its passage the planet would be said to have been in *transit across the sun's disc.* As the orbit of Venus does not lie in the plane of the ecliptic the phenomena of a transit of Venus are by no means so simple as the hypothetical transit just indicated. The inclination of the orbit of Venus to the ecliptic is $3°23'35''$, and it may therefore happen and indeed generally does happen that when Venus and the sun have the same geocentric longitude, the planet passes above the sun or below the sun and so a transit cannot occur unless the apparent distance of the planet from the suns centre is less than the suns apparent semi-diameter. But owing to the inclination of the orbit of Venus it may happen that even at a conjunction the apparent distance of the planet from the sun's centre may be many times as much as the suns apparent semi-diameter.

The geometrical relations of the sun, the earth and the planet at the time of a transit can be studied by supposing that the diameters of the earth and the planet are evanescent in comparison with the diameter of the sun so that the earth is represented by its centre E and Venus by its centre V.

If a transit is on the point of commencing or of ending the line EV should be a tangent to the solar globe. Its is therefore easy to see that the small angle expressing the heliocentric elongation of Venus from the earth at the moment of commencement of ending of a transit of Venus must be approximately $R(r - b)/br$ when R is the radius of the sun and r, b the respective distances of the earth and Venus from the sun. If we take the mean value of the suns apparent angular semi-diameter to be $16'$ and for r en b the values 1 and 0.72 respectively, we find that the required elongation as approximately $16' \times .28/.72 = 6'.2$. It thus appears that at a transit the heliocentric elongation of Venus from the earth must not exceed about $6'$. The conditions under which the transit takes place and its variations as seen from different points on the earths surface are so complicated as to require a

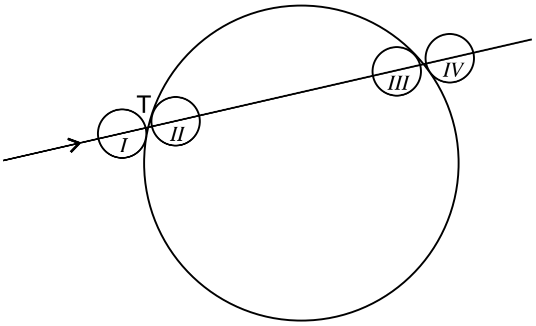

Figure 1: The internal and external contacts.

general investigation of the problem to which we now preceed and in which we shall regard the sun and the planet as exact spheres. When a transit of Venus is about to commence the circular disc of the planet, Fig. 1, comes into apparent contact with the circular disc of the sun. This initial stage of the phenomenon is known as the first external contact and is denoted by I. The planet then appears to enter slowly upon the disc of the sun, and in due course the next stage II known as *first internal contact* is reached. From this point the planet, now seen as a black disc on the brilliant background, advances across the sun's disc and after the lapse of four hours reaches the third critical stage III at what is known as *second internal contact.* Then the planet begins to pass off the suns disc, an finally arrives at IV or last external contact, and the phenomenon is at an end. As the external contacts cannot be observed so satisfactorily as the internal contacts the former are comparatively of small importance, and our attention will be devoted to the two internal contacts II and III.

To understand the geometrical problem involved in the transit of Venus we shall imagine a line drawn from the observer to T the point of apparent contact of the globes in stage II. It is evident that this line, though meeting both spheres, does not cut either of them. It must therefore be a common tangent to the two spheres. But such common tangent lines to the two spheres are generators of that common tangent cone which has its vertex exterior to the two spheres and hence we see that at the moments of II and III the observer must be situated at some point of that tangent cone. At the moments of external contact represented in I and IV the observer must be situated at some point on the other common tangent cone, namely that one which ha its vertex between the two spheres.

The theory of the transit of Venus must therefore be based on that of the common tangent cones to the two spheres which will be discussed in the next article.

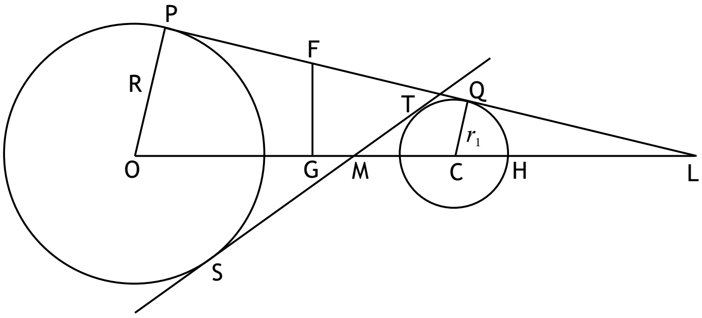

Figure 2: The tangent cones to the sun and planet.

2 Tangent cones to the sun and planet both regarded as spherical

Let O, C, Fig. 2, be the centres of the sun and the planet of which the radii are R, r_1 and let b denote OC. Then the double tangents PQ and ST meet OC in L and M the vertices of the common tangential cones to the two spheres.

Let x_1, y_1, z_1 be the coordinates of C with respect to three rectangular axes through the origin O. Then

$$x_1 R/(R \mp r_1),\ y_1 R/(R \mp r_1),\ z_1 R/(R \mp r_1)$$

with the lower signs are the coordinates of L and with the upper signs are the coordinates of M. If x, y, z are the coordinates of any point F on the cone with its vertex at L then the coordinates of any other point on the line PL will be obtained by assigning certain values to f and g in the expressions

$$\frac{fx + gx_1 R/(R - r_1)}{f + g},\ \frac{fy + gy_1 R/(R - r_1)}{f + g},\ \frac{fz + gz_1 R/(R - r_1)}{f + g}.$$

When the coordinates are substituted in the equation of either of the spheres they will give a quadratic in f/g corresponding to the two points in which FL meets that sphere. This equation for the sphere with centre O becomes

$$\begin{aligned} &f^2(x^2 + y^2 + z^2 - R^2)(R - r_1)^2 \\ &\quad + 2fgR(xx_1 + yy_1 + zz_1 - R^2 + Rr_1)(R - r_1) \\ &\quad + g^2 R^2\{b^2 - (R - r_1)^2\} = 0. \end{aligned}$$

Expressing the condition that this quadratic shall have equal roots because FL touches the sphere we find for the equation of the common tangent cone with vertex L

$$\begin{aligned} &(xx_1 + yy_1 + zz_1 - R^2 + Rr_1)^2 \\ &\quad = (x^2 + y^2 + z^2 - R^2)(b^2 - R^2 + 2Rr_1 - r_1^2). \end{aligned} \tag{1}$$

In like manner we obtain for the cone with its vertex at M

$$\begin{aligned}&(xx_1 + yy_1 + zz_1 - R^2 + Rr_1)^2\\&\qquad = (x^2 + y^2 + z^2 - R^2)(b^2 - R^2 - 2Rr_1 - r_1^2).\end{aligned} \tag{2}$$

Seen by an observer on cone (1) between Q and L the two circles appear on the same side of their common tangent. Seen by an observer on ST produced beyond T the two circles are on opposite sides of their common tangent.

3 Equation for determining the times of internal contact II and III

The transit of Mercury and Venus across the sun's disc must take place, as we have seen, when the planet is sufficiently near one of its nodes, the limits being $\sin^{-1}\{\cot i \tan D\}$ on either side of a node, where i is the inclination of the planet's orbit and D the semi-diameter of the sun. We shall suppose the planet to be near its *ascending* node on the ecliptic, and we shall confine our attention to the internal contact and obtain the equation by which they are determined.

The axes of reference and the symbols to be employed are as follows:

O is the origin of coordinates at the sun's centre.
$+X$ is from O towards ♈.
$+Y$ is from O towards the celestial point of long. 90° and lat. zero.
$+Z$ is from O towards the nole of the ecliptic.

The coordinates of the observer with respect to these axes are x, y, z and those of the centre of the planet are x_1, y_1, z_1.

O' is the centre of the earth. $O'X$, $O'Y$, $O'Z$ are the axes through O' parallel to OX, OY, OZ and x', y', z' are the coordinates with respect to $O'X$, $O'Y$, $O'Z$ of the point on the earth's surface occupied by the observer.
λ is the earth's heliocentric longitude.
r, b are the radii vectores from the sun to earth and Venus.
ρ is the distance from the centre of the earth to the observer.
ϕ is the geocentric latitude of the observer.
ϑ is the sidereal time on the meridian of the observer.
Ω is the longitude of the planet's ascending node.
ε is the inclination of the planet's orbit to the ecliptic.
θ is the angle round O swept over by the planet in its orbit since passing through its ascending node.

Equation (1) determines the times of interior contact of the planet if for

x, y, z, x', y', z', x_1, y_1, z_1 we substitute:

$$\left.\begin{aligned} x &= r\cos\lambda + x' \\ y &= r\sin\lambda + y' \\ z &= z'. \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} x' &= \rho\cos\phi\cos\vartheta \\ y' &= \rho\cos\omega\cos\phi\sin\vartheta + \rho\sin\omega\sin\phi \\ z' &= -\rho\sin\omega\cos\phi\sin\vartheta + \rho\cos\omega\sin\phi. \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} x_1 &= b\cos\Omega\cos\theta - b\sin\Omega\sin\theta\cos\varepsilon \\ y_1 &= b\sin\Omega\cos\theta + b\cos\Omega\sin\theta\cos\varepsilon \\ z_1 &= b\sin\theta\sin\varepsilon. \end{aligned}\right\} \tag{5}$$

These equations are obtained as follows:

The coordinates of O' are $r\cos\lambda$, $r\sin\lambda$, 0, and to find x, y, z we must add severally to the coordinates of O' the corresponding coordinates x', y', z' of the observer with respect to the parallel axes through O'.

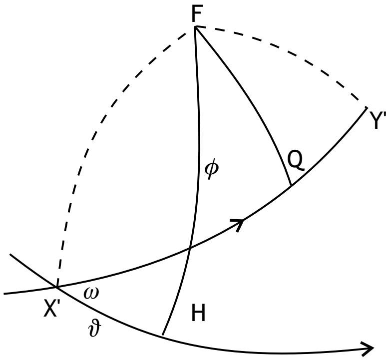

Figure 3: The coordinates of the observer relative to the axes through O'.

We may obtain equations (4) for x', y', z' from Fig. 3 in which F is the position of the observer, FH his meridian and $X'H$ the terrestrial equator. A plane through the earth's centre parallel to the ecliptic meets the earth's surface in $X'Y'$ and $X'Y' = 90°$. As $O'X'$ is parallel to $O\Upsilon$ the arc $X'H$ which is increasing by the earth's rotation must be the west hour angle of Υ for the meridian FH of the observer, that is, the local sidereal time ϑ. If FQ be perpendicular to $X'Y'$ the coordinates of the observer relative to the axes through O' are therefore

$$\begin{aligned} x' &= \rho\cos FX', \\ y' &= \rho\cos FY' &&= \rho\sin FX'\cos(FX'H - \omega), \\ z' &= \rho\sin FQ &&= \rho\sin FX'\sin(FX'H - \omega), \end{aligned}$$

and thus since

$$\sin FX' \cos FX'H = \cos\phi \sin\vartheta, \sin FX' \sin FX'H = \sin\phi,$$

and

$$\cos FX' = \cos\phi \cos\vartheta,$$

we see that x', y', z' have the values shown in (4).

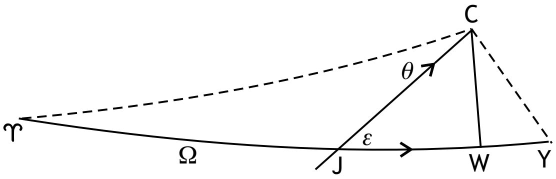

Figure 4: The coordinates of the planet relative to O.

In Fig. 4 C is the centre of the planet and J the ascending node if its orbit on the ecliptic ΥY. If CW be perpendicular to the ecliptic and the angle $\Upsilon Y = 90°$, the coordinates x_1, y_1, z_1 of C are $b \cos C\Upsilon$, $b \cos CY$, $b \sin CW$, and we obtain the values of x_1, y_1, z_1 given in (5).

4 Approximate solution of the general equation for internal contact

We are now to make the substitution indicated in the preceding article, and we have

$$\begin{aligned} xx_1 + yy_1 + zz_1 &= br\{\cos\theta\cos(\lambda - \Omega) + \cos\varepsilon \sin\theta \sin(\lambda - \Omega)\} \\ &\quad + x'x_1 + y'y_1 + z'z_1, \\ x^2 + y^2 + z^2 - R^2 &= r^2 + 2rx' \cos\lambda + 2ry' \sin\lambda + \rho^2 - R^2. \end{aligned}$$

We next avail ourselves of the fact that r_1^2/b^2, ρ^2/r^2, $\rho R^2/r^3$, and R^4/r^4, being respectively about

$$1/(18000)^2,\ 1/(23000)^2,\ 1/23000 \times 215^2,\ \text{and}\ 1/215^4,$$

are very small quantities (see *Table of elements of the solar system* at the end of the volume) and may be neglected as insensible. Thus we find approximately

$$\begin{aligned} (x^2 + y^2 + z^2 - R^2)^{\frac{1}{2}} &= r - R^2/2r + x' \cos\lambda + y' \sin\lambda, \\ \{b^2 - (R - r_1)^2\}^{\frac{1}{2}} &= b - R^2/2b + Rr_1/b. \end{aligned}$$

Making these substitutions, equation (1) becomes, after taking the square root of both sides and rejecting the negative sign for the radical because that relates only to the passage of the planet *behind* the sun,

$$\begin{aligned}&\cos\theta\cos(\lambda-\Omega)+\cos\varepsilon\sin\theta\sin(\lambda-\Omega)\\ &\quad=1-R^2(r-b)^2/2r^2b^2+Rr_1(r-b)/rb^2\\ &\qquad+(x'b\cos\lambda+y'b\sin\lambda-x'x_1-y'y_1-z'z_1)/rb.\end{aligned}\tag{6}$$

The time, which is the unknown quantity we are seeking, does not appear explicitly in the equation as at present written. It is, however, implicitly contained in the expressions for λ, θ, x', y', z', x_1, y_1, z_1, so that the equation appears to be of great complexity. But this complexity in unavoidable because the equation as it stands at present has to apply to transits of the planet for all time past and future. When we restrict our view to a single transit the equation admits of reduction to a manageable form giving all that is necessary for that particular transit.

We begin by considering the times at which the transit would commence and end if it could be viewed from the centre of the earth, in which case x', y', z' are all zero and the equation may be written

$$\begin{aligned}&\cos\theta\cos(\lambda-\Omega)+\cos\varepsilon\sin\theta\sin(\lambda-\Omega)\\ &\quad=1-R^2(r-b)^2/2r^2b^2+Rr_1(r-b)/rb^2.\end{aligned}\tag{7}$$

Each side of this equation expresses the cosine of the angle ψ subtended at the centre of the sun by the centres of Venus and the earth. If $\dot\theta$, $\dot\lambda$ be the known rates per hour at which the true anomalies of Venus and the earth are increasing, and if t_0 and t_1 be the Greenwich mean times at which the earth and Venus respectively arrive at the node, we have approximately

$$\theta=\dot\theta(t-t_1),\ \ \lambda-\Omega=\dot\lambda(t-t_0).$$

On the occasion of the most recent transit of Venus on December 6th, 1882, we had

$$r=0.9850,\ \ b=0.7205,\ \ R=0.004663,\ \ r_1=0.00004026,$$

when the mean distance of the earth from the sun is taken as unity. With these figures

$$\begin{aligned}R^2(r-b)^2/2r^2b^2&=0.000001510,\\ Rr_1(r-b)/rb^2&=0.000000097.\end{aligned}$$

The equation may therefore be written as follows:

$$\begin{aligned}\cos\psi\equiv\cos\dot\theta(t-t_1)\cos\dot\lambda(t-t_0)+\cos\varepsilon\sin\dot\theta(t-t_1)\sin\dot\lambda(t-t_0)\\ =1-0.000001413.\end{aligned}\tag{8}$$

Thus ψ is a small angle of $5'47''$, so that as ε is $3°23'31''$ it is easy to see that neither $\dot{\theta}(t-t_1)$ nor $\dot{\lambda}(t-t_0)$ can exceed $1°40'$. We may therefore express this equation with sufficient accuracy as follows:

$$\begin{aligned}1-\tfrac{1}{2}(t-t_1)^2\dot{\theta}^2-\tfrac{1}{2}(t-t_0)^2\dot{\lambda}^2+\dot{\theta}\dot{\lambda}(t-t_1)(t-t_0)\cos\varepsilon\\ =1-0.000001413,\end{aligned}$$

which gives a quadratic for t in which all the other quantities are known. When the substitutions for $\dot{\theta}$, $\dot{\lambda}$, ε, t_0, t_1 are made it is found that the roots of this equation are real, which shows that a transit takes place. If they were imaginary there would not be a transit. If they were equal then Venus would appear just to graze the sun's limb.

Suppose the real roots of the quadratic are t', t'' and that $t''>t'$. Then t' is the time at which the planet will appear to enter fully on the sun's disc (II) and at t'' the planet will begin to leave the disc (III). The duration of the transit is $t''-t'$. Thus the problem has been solved for the transit of Venus if it could be viewed from the centre of the earth.

5 On the application of the transit of Venus to the determination of the sun's distance

This application depends upon observations of the second and third contacts made from different stations, and we have first to obtain the theoretical expressions for such times of contact.

We see from equations (4) and (5) that we may write

$$\begin{aligned}x'&=\rho\alpha' &&\text{and} && x_1=b\alpha_1,\\ y'&=\rho\beta' &&\text{''} && y_1=b\beta_1,\\ z'&=\rho\gamma' &&\text{''} && z_1=b\gamma_1,\end{aligned}$$

where α', β', γ', α_1, β_1 and γ_1 are functions of the several angles ϕ, ϑ, ω, Ω, θ and ε and are independent of the linear quantities ρ and b.

Thus the last term in equation (6), viz:

$$(x'b\cos\lambda+y'b\sin\lambda-x'x_1-y'y_1-z'z_1)/rb,$$

becomes

$$(\alpha'\cos\lambda+\beta'\sin\lambda-\alpha'\alpha_1-\beta'\beta_1-\gamma'\gamma_1)\rho/r.$$

To obtain the times $t'+\Delta t'$ and $t''+\Delta t''$ of second and third contact, as seen by the observer whose terrestrial coordinates are $x'y'z'$, we first compute A' which is the value of

$$\alpha'\alpha_1+\beta'\beta_1+\gamma'\gamma_1-\alpha'\cos\lambda-\beta'\sin\lambda,$$

when the values of ϑ, θ and λ corresponding to the time t' have been introduced. In like manner A'' expresses the value of the same function corresponding to the time t''. Thus we have for the second contact

$$\cos\psi - \sin\psi \cdot \dot{\psi}\Delta t' = 1 - R^2(r-b)^2/2r^2b^2 + Rr_1(r-b)/rb^2 - A'\rho/r,$$

whence we obtain

$$\Delta t' = A'\rho/r\dot{\psi}\sin\psi,$$

and consequently the observer at x', y', z' sees second contact at the time

$$t' + A'\rho/r\dot{\psi}\sin\psi. \tag{9}$$

In like manner it is shown that for the same observer the time of third contact will be

$$t'' - A''\rho/r\dot{\psi}\sin\psi, \tag{10}$$

and accordingly for this observer the duration of the transit from second to third contact will be

$$t'' - t' - (A' + A'')\rho/r\dot{\psi}\sin\psi. \tag{11}$$

If the same transit be also observed from another station and if for this station B', B'' be the quantities corresponding to A', A'', then the duration of the transit as there seen will be

$$t'' - t' - (B' + B'')\rho/r\dot{\psi}\sin\psi. \tag{12}$$

Hence if D be the difference between the durations of the transit of the planet from second to third contact as seen from the two stations, we have

$$D = (B' + B'' - A' - A'')\rho/r\dot{\psi}\sin\psi. \tag{13}$$

In this equation A', A'', B', B'' are calculated by the formulae of §4. The angle ψ is given by the equation (8), and ψ is obtained by differentiation with regard to the time.

If finally D is determined by observation, then as ρ is known, r is found from equation (13). This is the famous method of determining the sun's distance proposed by Halley. It requires that both second and third contacts should be observed at each of the two stations.

There is also another method of deriving the distance of the sun from observations of the transit of Venus, which bears the name of its originator, De Lisle. This method has the advantage over Halley's that only two successful observations instead of four are needed and consequently the risks of failure by bad weather are correspondingly reduced.

Suppose that the times of observed second contact are obtained at two stations, then the interval will be from (9)

$$(t' + A'\rho/r\dot{\psi}\sin\psi) - (t' + B'\rho/r\dot{\psi}\sin\psi) = (A' - B')\rho/r\dot{\psi}\sin\psi.$$

If therefore this interval can be determined we shall have an equation for r.

Of course De Lisle's process can also be applied to a pair of observations of third contact made from two different stations.

The chief drawback to the transit of Venus as a method for the determination of the sun's distance arises from the difficulty of observing exactly the moment of contact between the disc of the planet and the limb of the sun. The movement of the planet is so slow and the limb of the sun so ill-defined that an uncertainty of several second is liable to be found in each observation.

From: Robert Ball, *A Treatise on Spherical Astronomy* (Cambridge, 1908), pp. 312–322